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Energy-momentum/Cotton tensor duality for AdS_4 black holes

Ioannis Bakas

*Department of Physics, University of Patras,
GR-26500 Patras, Greece
E-mail: bakas@ajax.physics.upatras.gr*

ABSTRACT: We consider the theory of gravitational quasi-normal modes for general linear perturbations of AdS_4 black holes. Special emphasis is placed on the effective Schrödinger problems for axial and polar perturbations that realize supersymmetric partner potential barriers on the half-line. Using the holographic renormalization method, we compute the energy-momentum tensor for perturbations satisfying arbitrary boundary conditions at spatial infinity and discuss some aspects of the problem in the hydrodynamic representation. It is also observed in this general framework that the energy-momentum tensor of black hole perturbations and the energy momentum tensor of the gravitational Chern-Simons action (known as Cotton tensor) exhibit an *axial-polar duality* with respect to appropriately chosen supersymmetric partner boundary conditions on the effective Schrödinger wavefunctions. This correspondence applies to perturbations of very large AdS_4 black holes with shear viscosity to entropy density ratio equal to $1/4\pi$, thus providing a dual graviton description of their hydrodynamic modes. We also entertain the idea that the purely dissipative modes of black hole hydrodynamics may admit Ricci flow description in the non-linear regime.

KEYWORDS: AdS-CFT Correspondence, Black Holes.

Contents

1. Introduction	1
2. Gravitational perturbations of AdS_4 black holes	4
2.1 Generalities	4
2.2 Axial (odd) perturbations	5
2.3 Polar (even) perturbations	7
2.4 Supersymmetric partner potentials	8
2.5 Asymptotic expansions at spatial infinity	9
2.6 The sign of $Im\omega$	12
3. Holographic energy-momentum tensor	13
3.1 General considerations	13
3.2 Static AdS_4 black holes	15
3.3 Axial perturbations	17
3.4 Polar perturbations	18
4. Hydrodynamic representation	20
4.1 First order hydrodynamics	20
4.2 Formal identifications	21
4.3 True hydrodynamic modes	23
5. Connection with the normalized Ricci flow on S^2	25
6. Energy-momentum/Cotton tensor duality	26
6.1 Chern-Simons gravitational action	27
6.2 The new correspondence for black holes	27
7. Conclusions	29
A. Coefficients of the asymptotic expansion	31
B. Energy-momentum tensor on ∂M_r	32

1. Introduction

The question of stability of the Schwarzschild metric against small perturbations of the geometry arose more than half century ago in the seminal work of Regge and Wheeler, [1]. Since then, the subject has grown enormously (see, for instance, the selected works [2–4],

among many other important contributions) and developed into what has become known as the theory of quasi-normal modes (for reviews of the mathematical and physical aspects of the problem see, for instance, references [5] and [6], respectively). In recent years, the theory of quasi-normal modes has also been extended to black-hole solutions in space-times with cosmological constant Λ , and in particular to the AdS_4 Schwarzschild background, [7, 8], which is the subject of this work.

Although the theory of gravitational perturbations of black holes can be studied systematically in higher dimensions as well, it is important to realize that four space-time dimensions are rather special in this framework, since they exhibit a remarkable duality among the two distinct classes of perturbations. The duality exists irrespective of Λ and connects the effective Schrödinger problems that describe the axial and polar perturbations of the metric. This relation was first discovered more than thirty years ago, [4] (but see also reference [5] for an extensive presentation), by considering gravitational perturbations of the Schwarzschild metric (without cosmological constant) and it gave rise to what has become known later in the literature as partner potentials in supersymmetric quantum mechanics, [9, 10]. The axial-polar relation persists in the presence of cosmological constant, [7, 8], although supersymmetric quantum mechanics does not necessarily respect the boundary conditions imposed on the effective wave-functions at spatial infinity. Yet, there is no fundamental explanation of this occurrence, to the best of our knowledge, and any new insight into the problem is certainly welcome. Furthermore, there could be reformulations and/or different manifestations of this duality in areas where the general theory of quasi-normal modes is applicable.

AdS/CFT correspondence, [11–13], and in particular its generalization to finite temperature field theory, [14], provide such a framework using the AdS Schwarzschild solution as background geometry on the bulk. In fact, many well known facts about the thermodynamics of AdS black holes, as they were originally formulated by Hawking and Page, [15], found a natural manifestation in AdS/CFT correspondence, [14]. It was subsequently realized that the theory of quasi-normal modes also had a natural place in this framework, as it describes small deviations from the equilibrium state in finite temperature field theory, [16]; the inverse time scale for return to equilibrium is given by the (minus) imaginary part of the corresponding quasi-normal mode. Although scalar field perturbations on AdS Schwarzschild backgrounds were in focus at first, the holographic description of the gravitational quasi-normal modes were also investigated and led to some important developments. The calculations are based on the method of holographic renormalization, [17–21], which under the appropriate boundary conditions at spatial infinity yields the energy-momentum tensor of the gravitational background on the bulk; this method puts on firm ground a previous proposal for the definition of quasi-local energy in gravitational theories, [22], and overcomes its limitations. One of the most spectacular results derived in this context in recent years has been the connection between black holes and relativistic hydrodynamics and, in particular, the derivation of a universal value for the ratio of shear viscosity to entropy density, known as KSS bound, [23, 24]. They complement quite nicely the old ideas on black hole hydrodynamics that led to the membrane paradigm, [25].

AdS_4/CFT_3 correspondence is less studied in the literature up to this date and some

new features in the holographic description of four-dimensional gravity may arise. As far as the previous discussion is concerned, the axial-polar duality among the gravitational perturbations of AdS_4 black holes may have an interesting manifestation in the three-dimensional field theory at the conformal boundary of space-time. Hopefully, it may also help to explain in more fundamental terms why there is an underlying supersymmetric quantum mechanics in the mathematical description of the gravitational quasi-normal modes of four-dimensional black holes. Here, we present some new results in this direction and reformulate (at least part of) the problem as black hole energy-momentum tensor/Cotton tensor duality using general boundary conditions on the wave-functions of the effective Schrödinger problems for axial and polar perturbations. In this way, the gravitational Chern-Simons action on the dual conformal boundary comes into play, since its energy momentum tensor is by definition the Cotton tensor in three dimensions. Further details and applications of the correspondence will be presented elsewhere.

The main material of this paper is based on the theory of quasi-normal modes and the method of holographic renormalization for computing the boundary energy-momentum tensor. Section 2 contains an overview of the gravitational perturbations of black holes in four space-time dimensions with emphasis on the AdS_4 Schwarzschild background. We will not include the results of numerical investigations that have been carried out in detail and appear in several research and review papers. We derive, however, the asymptotic expansion of the metric perturbations at spatial infinity that will be useful in the calculations. Section 3 contains an account of the holographic computation of the energy-momentum tensor in four-dimensional linearized Einstein gravity and then proceeds with its evaluation under general boundary conditions on the wave-functions of the effective Schrödinger equations. Some intermediate steps of the calculations described in sections 2 and 3 are given in appendices A and B, respectively. Section 4 contains some connections with the hydrodynamic representation of black hole perturbations, while keeping the presentation superficially simple, and selects a privileged set of boundary conditions by requiring that the shear viscosity of axial and polar perturbations to be equal. Section 5 contains as side remark the idea that the pure dissipative hydrodynamic modes of black hole physics may be accounted by the normalized Ricci flow (when suitably embedded into Einstein's equations with negative cosmological constant) at the non-linear level. Section 6 contains our main observations on the boundary manifestation of axial-polar duality based on the general formulae included in this paper. It is also shown the this duality operates entirely within the KSS bound for the ratio of shear viscosity to the entropy density of black holes, thus providing a correspondence between black hole hydrodynamics and the gravitational Chern-Simons theory. Section 7 contains our conclusions and a small list of selected directions for future work.

Throughout this paper, we set $8\pi G = \kappa^2$ and Newton's constant is normalized as $G = 1$ in the Schwarzschild metric. The boundary conditions (Dirichlet or mixed) always refer to the wave-functions of the effective Schrödinger problems at spatial infinity and not to the metric perturbations themselves; of course one follows from the other. We will also abuse the term “supersymmetric quantum mechanics”, since there are no fermions here. The term “spatial infinity” will always refer to $r = \infty$ in the radial direction of space-time. Finally,

the perturbations of the metric (and related geometric quantities) are complex for each quasi-normal mode. Apparently, real expressions will arise by appropriate superposition, although it is not yet known (as far as we can tell from the literature) whether these modes form a complete set in the strict mathematical sense.

2. Gravitational perturbations of AdS_4 black holes

In this section we review the basic features of linear perturbations around the four dimensional Schwarzschild background,

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + \delta g_{\mu\nu}, \tag{2.1}$$

using the canonical decomposition of $\delta g_{\mu\nu}$ into two distinct classes called axial and polar perturbations, [1]. The resulting theory of quasi-normal modes is formulated in the presence of cosmological constant and some special features of AdS_4 black holes are discussed in detail (see also [7, 8]). Hopefully, the present exposition can be of more general value to the interested reader, as it contains a number of explicit results together with the companion appendix A.

2.1 Generalities

First, we recall for notational purposes some basic facts about black holes that will be used throughout this paper.

Einstein equations in four space-time dimensions with cosmological constant Λ ,

$$R_{\mu\nu} = \Lambda g_{\mu\nu}, \tag{2.2}$$

admit the Schwarzschild solution as spherically symmetric static configuration of the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \tag{2.3}$$

with

$$f(r) = 1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2 \tag{2.4}$$

having the appropriate asymptotic behavior fixed by Λ .

The Schwarzschild radius of AdS_4 black holes is provided by the real root of $f(r) = 0$ occurring at

$$r_h = \frac{1}{\sqrt{-\Lambda}} \left[\left(\sqrt{1 - 9m^2\Lambda} + 3m\sqrt{-\Lambda} \right)^{1/3} - \left(\sqrt{1 - 9m^2\Lambda} - 3m\sqrt{-\Lambda} \right)^{1/3} \right]. \tag{2.5}$$

Thus, the black hole radius takes values $0 < r_h < 2m$ depending on the size of Λ . When Λ approaches zero, r_h tends to $2m$, whereas for $\Lambda \ll 0$, r_h comes close to 0.

It is also useful to introduce the tortoise coordinate r_* which is defined by

$$dr_* = \frac{dr}{f(r)}. \tag{2.6}$$

When $\Lambda = 0$, r_* ranges from $-\infty$ to $+\infty$, as r ranges from the black hole horizon located at $r = r_h$ to infinity. But when $\Lambda < 0$, which is of interest here, r_* ranges from $-\infty$ up to some finite value that can be set equal to zero by appropriate choice of the integration constant. For AdS_4 black holes, in particular, we have explicitly

$$r_* = \frac{r_h}{4(r_h - 3m)} \left(r_h \log \frac{(2r + r_h)^2 + a^2}{4(r - r_h)^2} + 2a \frac{r_h - 6m}{r_h + 6m} \left[\arctan \frac{2r + r_h}{a} - \frac{\pi}{2} \right] \right) \quad (2.7)$$

setting for convenience

$$a = \sqrt{-\frac{3}{\Lambda} \left(1 + \frac{6m}{r_h} \right)}. \quad (2.8)$$

AdS black holes come in different sizes and their thermodynamic properties depend crucially on the magnitude of r_h relative to the AdS radius

$$L = \sqrt{-\frac{3}{\Lambda}}. \quad (2.9)$$

Large black holes have $r_h > L$ and become the dominant configurations at high temperatures, whereas small black holes have $r_h < L$ and they are always unstable to decay either into pure thermal radiation or to black holes with larger mass. In general we have the following relation among the parameters of the AdS_4 Schwarzschild background

$$m - r_h = \frac{1}{2L^2} r_h (r_h^2 - L^2). \quad (2.10)$$

Thus, large black holes have $r_h < m$, whereas small black holes have $r_h > m$.

Finally, we recall that very large black holes are naturally associated to the limit $r_h \rightarrow \infty$, in which case $f(r)$ is replaced by

$$f(r) = -\frac{2m}{r} - \frac{\Lambda}{3} r^2 \quad (2.11)$$

by dropping the constant term. Then, the black holes become essentially flat and their horizon is related to the other parameters by the simple expression

$$r_h^3 = -\frac{6m}{\Lambda}. \quad (2.12)$$

2.2 Axial (odd) perturbations

The first class of metric perturbations of four dimensional black holes is tabulated by matrices labeled by (t, r, θ, ϕ) of the following form

$$\delta g_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & h_0(r) \\ 0 & 0 & 0 & h_1(r) \\ 0 & 0 & 0 & 0 \\ h_0(r) & h_1(r) & 0 & 0 \end{pmatrix} e^{-i\omega t} \sin\theta \partial_\theta P_l(\cos\theta), \quad (2.13)$$

using the Legendre polynomials $P_l(\cos\theta)$. More general expressions in terms of spherical harmonics $Y_l^m(\theta, \phi)$ can also be employed, but one may only use axially symmetric perturbations, setting $m = 0$ without loss of generality. Axial perturbations correspond to the so called vector sector or shear channel in the dictionary of AdS/CFT correspondence.

The linear gravitational perturbations $\delta R_{\mu\nu} = \Lambda \delta g_{\mu\nu}$ about the Schwarzschild background yield the following equation for the $(\theta\phi)$ -component,

$$h_0(r) = i \frac{f(r)}{\omega} (f(r)h_1(r))', \quad (2.14)$$

whereas the equation for the $(r\phi)$ -component reads

$$\frac{2}{r}h_0(r) - h_0'(r) = i \frac{f(r)}{\omega} \left(\frac{\omega^2}{f(r)} - \frac{(l-1)(l+2)}{r^2} \right) h_1(r). \quad (2.15)$$

These form a coupled system of first order differential equations for the unknown functions $h_0(r)$ and $h_1(r)$, which are otherwise unrelated. The $(t\phi)$ -component gives rise to a second order differential equation, which, however, is trivially satisfied by virtue of the previous two equations. All other components of $\delta R_{\mu\nu}$ are identically zero and yield no further conditions.

Following Regge and Wheeler, [1], we define the following variable

$$\Psi_{\text{RW}}(r) = \frac{f(r)}{r} h_1(r), \quad (2.16)$$

which turns out to satisfy the effective one-dimensional Schrödinger equation

$$\left(-\frac{d^2}{dr_\star^2} + V_{\text{RW}}(r) \right) \Psi_{\text{RW}}(r) = \omega^2 \Psi_{\text{RW}}(r) \quad (2.17)$$

with respect to the tortoise coordinate r_\star with potential

$$V_{\text{RW}}(r) = f(r) \left(\frac{l(l+1)}{r^2} - \frac{6m}{r^3} \right). \quad (2.18)$$

Thus, one is led to consider solutions of the Regge-Wheeler-Schrödinger problem by imposing appropriate boundary conditions (typically ingoing at the black hole horizon and outgoing at spatial infinity), which in turn can determine $\Psi_{\text{RW}}(r)$ (and hence $h_1(r)$ and subsequently $h_0(r)$) together with the allowed spectrum of quasi-normal mode frequencies ω .

V_{RW} depend on l and represent spherically symmetric potentials surrounding the black hole. Plotting the potentials as function of the tortoise radial coordinate can only be made numerically because r cannot be expressed in terms of r_\star in closed form. For $\Lambda = 0$ the potentials are manifestly positive everywhere and extend on the real line $-\infty < r_\star < \infty$ falling off to zero at both ends. For $\Lambda < 0$, on the other hand, the potentials extend on the half-line $-\infty < r_\star \leq 0$, becoming zero on the horizon and reaching a finite positive value at spatial infinity. In this case, however, V_{RW} are not always everywhere positive definite, but they can become negative for sufficiently low values of l , namely for large black holes with

$$\frac{m}{r_h} > \frac{l(l+1)}{6}, \quad (2.19)$$

thus exhibiting a laguna. Although the differences between large and small AdS_4 black holes leave their footprints on the shape of the effective potential barriers for sufficiently small values of l , their plots are alike for large values of l exhibiting a maximum peak followed by a local minimum as r increases towards spatial infinity.

2.3 Polar (even) perturbations

This is a complementary class of metric perturbations parametrized by four arbitrary radial functions of the general form

$$\delta g_{\mu\nu} = \begin{pmatrix} f(r)H_0(r) & H_1(r) & 0 & 0 \\ H_1(r) & H_2(r)/f(r) & 0 & 0 \\ 0 & 0 & r^2K(r) & 0 \\ 0 & 0 & 0 & r^2K(r)\sin^2\theta \end{pmatrix} e^{-i\omega t} P_l(\cos\theta) \quad (2.20)$$

They correspond to the so called scalar sector or sound channel in the dictionary of AdS/CFT correspondence. Study of such linear perturbations $\delta R_{\mu\nu} = \Lambda g_{\mu\nu}$ about the Schwarzschild background yields

$$H_0(r) = H_2(r) . \quad (2.21)$$

This choice will be made from the beginning to simplify the remaining equations.

Tedious computation shows that the (tr) - $(r\theta)$ - and $(t\theta)$ -components of the perturbation yield the following equations, respectively,

$$rK'(r) + \left(1 - \frac{rf'(r)}{2f(r)}\right) K(r) - H_0(r) - i\frac{l(l+1)}{2\omega r} H_1(r) = 0 , \quad (2.22)$$

$$(f(r)H_0(r))' - f(r)K'(r) + i\omega H_1(r) = 0 , \quad (2.23)$$

$$(f(r)H_1(r))' + i\omega (H_0(r) + K(r)) = 0 . \quad (2.24)$$

Together they form a coupled system of first order differential equations for the three unknown functions $H_0(r)$, $H_1(r)$ and $K(r)$. The other components of the perturbation either yield second order equations or else $\delta R_{\mu\nu}$ vanishes identically. Note here, however, that there is an additional algebraic condition among the three radial functions

$$(2f(r) - rf'(r) - l(l+1)) H_0(r) + \frac{i}{2\omega} (4\omega^2 r - l(l+1)f'(r)) H_1(r) = \left(2f(r) + rf'(r) - l(l+1) + 2\Lambda r^2 + \frac{r^2}{2f(r)} (4\omega^2 + f'^2(r))\right) K(r) , \quad (2.25)$$

which follows from consistency of the various second order equations with the first order system above; it can also be viewed as integral of the first order system above.

Following Zerilli, [2], we now define the following variable

$$\Psi_Z(r) = \frac{r^2}{(l-1)(l+2)r + 6m} \left(K(r) - i\frac{f(r)}{\omega r} H_1(r) \right) , \quad (2.26)$$

which turns out to satisfy an effective Schrödinger equation, as before,

$$\left(-\frac{d^2}{dr_\star^2} + V_Z(r)\right) \Psi_Z(r) = \omega^2 \Psi_Z(r) \quad (2.27)$$

with different potential,

$$V_Z(r) = \frac{f(r)}{[(l-1)(l+2)r + 6m]^2} \left(l(l+1)(l-1)^2(l+2)^2 - 24m^2\Lambda \right. \\ \left. + \frac{6m}{r}(l-1)^2(l+2)^2 + \frac{36m^2}{r^2}(l-1)(l+2) + \frac{72m^3}{r^3} \right). \quad (2.28)$$

Again, one has to find solutions and determine the quasi-normal mode spectrum under appropriate boundary conditions, as before. This will, in turn, lead to expressions for the three unknown radial functions of polar perturbations.

As before, V_Z depend on l and represent spherically symmetric potential barriers surrounding the black hole, which are always positive definite reaching a finite value at spatial infinity. For $\Lambda < 0$, the shape of the potential depends on the size of the black hole. In fact, V_Z appear to increase monotonically for large black holes with sufficiently low values of l , whereas for large values of l they exhibit a maximum peak followed by a local minimum as r increases towards spatial infinity. In these cases, V_Z resemble the shape of V_{RW} , but they rise higher than them for given l .

2.4 Supersymmetric partner potentials

The Regge-Wheeler and Zerilli potentials admit the following representation

$$V_{RW}(r) = W^2(r) - \frac{dW(r)}{dr_\star} + \omega_s^2 \quad (2.29)$$

and

$$V_Z(r) = W^2(r) + \frac{dW(r)}{dr_\star} + \omega_s^2 \quad (2.30)$$

in terms of a suitably chosen real (positive) function, [4, 5, 7] (but see also [8])

$$W(r) = \frac{6mf(r)}{r[(l-1)(l+2)r + 6m]} + i\omega_s, \quad (2.31)$$

setting

$$\omega_s = -\frac{i}{12m}(l-1)l(l+1)(l+2). \quad (2.32)$$

Thus, the two Schrödinger problems under investigation take the closely related form

$$\left(-\frac{d^2}{dr_\star^2} + W^2 \mp \frac{dW}{dr_\star}\right) \Psi(r_\star) = (\omega^2 - \omega_s^2) \Psi(r_\star) \quad (2.33)$$

and resemble supersymmetric partner potentials generated by the superpotential $W(r_\star)$, [4, 5, 7]. The quantity $E = \omega^2 - \omega_s^2$ serves as the energy of the effective quantum mechanical problem, but unlike conventional supersymmetric quantum mechanics, [9, 10], it is not bounded below by zero. In fact, due to the physical boundary conditions imposed on the

wave functions associated to perturbations of black holes, the quantum theory is that of an open system and the energies (and hence ω) are in general complex.

Due to this relation, which is only present in four space-time dimensions, the solutions are inter-connected using the conjugate pair of first order operators

$$A = \frac{d}{dr_\star} + W(r_\star), \quad A^\dagger = -\frac{d}{dr_\star} + W(r_\star). \quad (2.34)$$

The two effective Hamiltonians are simply written as

$$H_{\text{RW}} = A^\dagger A + \omega_s^2, \quad H_Z = AA^\dagger + \omega_s^2. \quad (2.35)$$

Then, if $\Psi_{\text{RW}}(r_\star)$ is a solution of the Regge-Wheeler equation with frequency ω , the function

$$A\Psi_{\text{RW}}(r_\star) = i(\omega_s - \omega)\Psi_Z(r_\star) \quad (2.36)$$

will be solution of the Zerilli equation with the same frequency. Likewise, a solution of the Zerilli equation with frequency ω gives rise to solution of the Regge-Wheeler equation with the same frequency, as

$$A^\dagger\Psi_Z(r_\star) = i(\omega_s + \omega)\Psi_{\text{RW}}(r_\star). \quad (2.37)$$

These relations are particularly useful for justifying mixed boundary conditions on the wave functions. For $\Lambda < 0$, one typically imposes perfectly reflecting Dirichlet boundary conditions on the wave functions at spatial infinity located at $r_\star = 0$. If the axial and polar perturbations satisfy simultaneously

$$\Psi_{\text{RW}}(r_\star = 0) = 0 = \Psi_Z(r_\star = 0) \quad (2.38)$$

supersymmetric quantum mechanics will also imply the Neumann boundary conditions

$$\frac{d}{dr_\star}\Psi_{\text{RW}}(r_\star) \Big|_{r_\star=0} = 0 = \frac{d}{dr_\star}\Psi_Z(r_\star) \Big|_{r_\star=0}, \quad (2.39)$$

which are too restrictive to hold all together. The conflict is resolved either by abandoning supersymmetry, meaning that the spectrum of quasi-normal modes of axial and polar perturbations is taken to be different, or by imposing mixed boundary conditions as dictated by equations (2.36) and (2.37) above.

We also note for completeness that when $\Lambda = 0$ the boundary conditions imposed at spatial infinity, namely outgoing waves for either axial or polar perturbations, are compatible with supersymmetric quantum mechanics.

2.5 Asymptotic expansions at spatial infinity

The wave functions $\Psi_{\text{RW}}(r)$ and $\Psi_Z(r)$ represent incoming waves to the black-hole. Therefore, they have the following asymptotic expansion close to the horizon, following closely the analysis of reference [16],

$$\Psi(r) = \sum_{n=0}^{\infty} a_n \left(1 - \frac{r_h}{r}\right)^n e^{-i\omega r_\star}. \quad (2.40)$$

In either case, the coefficients a_n depend upon ω . They satisfy a three-term recursion relation when substituted into the Regge-Wheeler equation and a five-term recursion relation when substituted into the Zerilli equation. Both power series expansions make good sense for all r when $\Lambda < 0$, because their radius of convergence extends to infinity. Then, the boundary conditions at spatial infinity impose additional constraints on the coefficients a_n , which in turn determine the spectrum of allowed quasi-normal modes of AdS_4 black holes by numerical methods.

Here, we will reorganize the series expansion of the wave functions in powers of $1/r$ to provide their asymptotic behavior at spatial infinity for $\Lambda < 0$. The coefficients will be determined up to the order relevant for the computation of the energy-momentum tensor for axial and polar perturbations of AdS_4 black holes. Thus, these coefficients will be obtained under general boundary conditions, but in the applications specific choices will be made at spatial infinity.

(i) Axial perturbations: the asymptotic expansion of the Regge-Wheeler wave function at spatial infinity is taken to be

$$\Psi_{\text{RW}}(r) = \left(I_0 + \frac{I_1}{r} + \frac{I_2}{r^2} + \frac{I_3}{r^3} + \dots \right) e^{-i\omega r_*}, \quad (2.41)$$

where the coefficients I_k depend upon ω and they are determined recursively from I_0 and I_1 as

$$\frac{2\Lambda}{3} I_2 = 2i\omega I_1 - l(l+1)I_0, \quad (2.42)$$

$$2\Lambda I_3 = 4i\omega I_2 - (l-1)(l+2)I_1 + 6mI_0 \quad (2.43)$$

and so on. The boundary conditions at spatial infinity are solely expressed in terms of I_0 and I_1 .

The asymptotic expansion of the metric functions $h_0(r)$ and $h_1(r)$ near spatial infinity are given in all generality by

$$h_0(r) = \left(\alpha_0 r^2 + \beta_0 r + \gamma_0 + \frac{\delta_0}{r} + \dots \right) e^{-i\omega r_*}, \quad (2.44)$$

$$h_1(r) = \left(\frac{\alpha_1}{r} + \frac{\beta_1}{r^2} + \dots \right) e^{-i\omega r_*}, \quad (2.45)$$

where the coefficients can be found in appendix A after expressing them in terms of I_0 and I_1 for convenience.

(ii) Polar perturbations: likewise, the asymptotic expansion of the Zerilli wave function at spatial infinity is taken to be

$$\Psi_{\text{Z}}(r) = \left(J_0 + \frac{J_1}{r} + \frac{J_2}{r^2} + \frac{J_3}{r^3} + \frac{J_4}{r^4} + \dots \right) e^{-i\omega r_*}, \quad (2.46)$$

where J_k depend upon ω and they are determined recursively from J_0 and J_1 via the relations

$$\frac{2\Lambda}{3}J_2 = 2i\omega J_1 - \left(l(l+1) - \frac{24m^2\Lambda}{(l-1)^2(l+2)^2} \right) J_0, \quad (2.47)$$

$$2\Lambda J_3 = 4i\omega J_2 - \left((l-1)(l+2) - \frac{24m^2\Lambda}{(l-1)^2(l+2)^2} \right) J_1 + \frac{6m}{(l-1)(l+2)} \left(l(l+1) + 2 - \frac{48m^2\Lambda}{(l-1)^2(l+2)^2} \right) J_0, \quad (2.48)$$

$$4\Lambda J_4 = 6i\omega J_3 - \left(l(l+1) - 6 - \frac{24m^2\Lambda}{(l-1)^2(l+2)^2} \right) J_2 + \frac{24m}{(l-1)(l+2)} \left(1 - \frac{12m^2\Lambda}{(l-1)^2(l+2)^2} \right) J_1 - \frac{72m^2}{(l-1)^2(l+2)^2} \left(l(l+1) + 1 - \frac{36m^2\Lambda}{(l-1)^2(l+2)^2} \right) J_0 \quad (2.49)$$

and so on. The boundary conditions at spatial infinity are solely expressed in terms of J_0 and J_1 , in analogy with the axial perturbations.

The asymptotic expansion of the metric functions $H_0(r)$, $H_1(r)$ and $K(r)$ take the following form at spatial infinity,

$$H_0(r) = \frac{3m}{(l-1)(l+2)} \left(\frac{A_0}{r} + \frac{B_0}{r^2} + \frac{C_0}{r^3} + \dots \right) e^{-i\omega r_*}, \quad (2.50)$$

$$H_1(r) = -\frac{3i\omega}{\Lambda} \left(\frac{A_1}{r} + \frac{B_1}{r^2} + \frac{C_1}{r^3} + \dots \right) e^{-i\omega r_*}, \quad (2.51)$$

$$K(r) = \left(R + \frac{A}{r} + \frac{B}{r^2} + \frac{C}{r^3} + \dots \right) e^{-i\omega r_*}. \quad (2.52)$$

The computations are much more involved now and the various coefficients are given explicitly in appendix A expressing them in terms of J_0 and J_1 alone.

These expansions are consistent with the differential equations as well as with the algebraic constraint satisfied by the metric functions of axial and polar perturbations with generic boundary conditions. We also note for completeness that the wave functions could have been expanded differently, e.g.,

$$\Psi_{\text{RW}}(r) = \left(I'_0 + \frac{I'_1}{r} + \frac{I'_2}{r^2} + \dots \right) e^{i\omega r_*}, \quad (2.53)$$

resembling the form of outgoing (rather than incoming) waves at spatial infinity. The two expressions are equivalent provided that $I'_0 = I_0$, $I'_1 = I_1 - (6i\omega/\Lambda)I_0$, etc, using the asymptotic expansion $r_* = 3/(\Lambda r) + \dots$. Similar remarks apply to the expansion of $\Psi_Z(r)$ at spatial infinity.

(iii) Supersymmetric partner boundary conditions: if the spectrum of axial and polar perturbations are related by supersymmetric quantum mechanics, one should adopt mixed boundary conditions at spatial infinity that are consistent with the general relation

$$\left(-\frac{d}{dr_*} + W(r_*) \right) \Psi_Z(r_*) = i(\omega_s + \omega) \Psi_{\text{RW}}(r_*). \quad (2.54)$$

Then, the coefficients J_0 and J_1 arising in the asymptotic expansion of $\Psi_Z(r)$ should be related to the corresponding coefficients I_0 and I_1 of $\Psi_{RW}(r)$ as

$$i(\omega_s - \omega)J_0 = \frac{\Lambda}{3}I_1 + \left(i(\omega_s - \omega) - \frac{2m\Lambda}{(l-1)(l+2)} \right) I_0, \quad (2.55)$$

$$i(\omega_s - \omega)J_1 = \left(i(\omega_s + \omega) - \frac{2m\Lambda}{(l-1)(l+2)} \right) I_1 - \left(l(l+1) - \frac{12m^2\Lambda}{(l-1)^2(l+2)^2} \right) I_0, \quad (2.56)$$

Conversely, we also have

$$i(\omega_s + \omega)I_0 = -\frac{\Lambda}{3}J_1 + \left(i(\omega_s + \omega) - \frac{2m\Lambda}{(l-1)(l+2)} \right) J_0, \quad (2.57)$$

$$i(\omega_s + \omega)I_1 = \left(i(\omega_s - \omega) - \frac{2m\Lambda}{(l-1)(l+2)} \right) J_1 + \left(l(l+1) - \frac{12m^2\Lambda}{(l-1)^2(l+2)^2} \right) J_0. \quad (2.58)$$

The simplest possibility of this kind is to impose Dirichlet boundary condition on the axial perturbations and mixed boundary condition on the polar perturbations, so that the coefficients are fixed by the relations

$$I_0 = 0, \quad I_1 = \frac{3i}{\Lambda}(\omega_s - \omega)J_0, \quad J_1 = \frac{3}{\Lambda} \left(i(\omega_s + \omega) - \frac{2m\Lambda}{(l-1)(l+2)} \right) J_0. \quad (2.59)$$

Other special choices of boundary conditions will be made later.

2.6 The sign of $Im\omega$

The perturbations diminish at late times provided that $Im\omega < 0$, in which case

$$\tau = -\frac{1}{Im\omega} \quad (2.60)$$

provides the characteristic time scale for return to equilibrium. Otherwise, the perturbations grow large at spatial infinity and stability is at stake; also, in such cases, the linear approximation can not be reliably used to extract the late time behavior of the energy-momentum tensor on the boundary. The sign of $Im\omega$ can be shown to be negative when perfectly reflecting boundary conditions are imposed at spatial infinity for the polar perturbations and the same is true for the axial perturbations, at least in those cases that the Regge-Wheeler potential does not form a laguna. Different boundary conditions at spatial infinity may affect the sign of $Im\omega$ for some modes in the spectrum, but there is no general proof that this is indeed the case.

The standard analytic argument to address this question, [16] (but see also [7]), and which is extended here to general boundary conditions, starts with the observation that the substitution $\Psi(r) = u(r)\exp(-i\omega r_*)$ yields the differential equation

$$\frac{d}{dr} \left(f(r) \frac{du(r)}{dr} \right) - 2i\omega \frac{du(r)}{dr} - \frac{V(r)}{f(r)} u(r) = 0 \quad (2.61)$$

for either Regge-Wheeler or Zerilli potentials. Multiplying it with the complex conjugate function $\bar{u}(r)$ and integrating over r , we obtain (after integrating by parts the first term)

the following relation,

$$\int_{r_h}^{\infty} dr \left(f(r) \left| \frac{du(r)}{dr} \right|^2 + 2i\omega\bar{u}(r) \frac{du(r)}{dr} + \frac{V(r)}{f(r)} |u(r)|^2 \right) = f(r)\bar{u}(r) \frac{du(r)}{dr} \Big|_{r=\infty} . \quad (2.62)$$

The right-hand side is simply $(\Lambda/3)J_1\bar{J}_0$ (respectively $(\Lambda/3)I_1\bar{I}_0$) for polar (respectively axial) perturbations satisfying general boundary conditions. Taking the imaginary part of this integral equation, let us say for polar perturbations, and integrating by parts the complex conjugate term, we obtain

$$2i \operatorname{Im}\omega \int_{r_h}^{\infty} dr \bar{u}(r) \frac{du(r)}{dr} = \frac{\Lambda}{3} \operatorname{Im}(J_1\bar{J}_0) - \bar{\omega} (|J_0|^2 - |u(r_h)|^2) . \quad (2.63)$$

This expression yields, upon substitution into the original equation, the final result

$$\int_{r_h}^{\infty} dr \left(f(r) \left| \frac{du(r)}{dr} \right|^2 + \frac{V_Z(r)}{f(r)} |u(r)|^2 \right) = \frac{\Lambda}{3} J_1\bar{J}_0 - \frac{\Lambda}{3} \operatorname{Im}(J_1\bar{J}_0) \frac{\omega}{\operatorname{Im}\omega} + \frac{|\omega|^2}{\operatorname{Im}\omega} (|J_0|^2 - |u(r_h)|^2) . \quad (2.64)$$

The left-hand side is positive definite and therefore $\operatorname{Im}\omega < 0$ when $J_0 = 0$. This argument is certainly inconclusive for more general boundary conditions. For axial perturbations the coefficients are replaced by I_0 and I_1 but the corresponding potential V_{RW} can become negative for large black-holes when l is sufficiently low. Thus, in those cases, the analytic argument becomes inconclusive, even for $I_0 = 0$, but numerical analysis shows that all such modes have $\operatorname{Im}\omega < 0$.

In the following, we will assume that the spectrum of quasi-normal modes have negative imaginary part either by selecting boundary conditions for which this is manifest, by the argument above, or by employing numerical methods to pin down those boundary conditions that yield $\operatorname{Im}\omega < 0$.

3. Holographic energy-momentum tensor

In this section, we briefly review the construction of the boundary energy-momentum tensor for AdS gravity, following [20], and apply it to the Schwarzschild solution and its perturbations. The same results can be obtained using Fefferman-Graham coordinates, as in the more systematic analysis of holographic renormalization presented in references [17–19, 21]. The companion appendix B summarizes the results of intermediate steps in the calculation.

3.1 General considerations

According to the AdS/CFT correspondence, the vacuum expectation value of the energy-momentum tensor of the boundary quantum field theory,

$$\langle T^{ab} \rangle = \frac{2}{\sqrt{-\det\gamma}} \frac{\delta S_{\text{eff}}}{\delta\gamma_{ab}} , \quad (3.1)$$

is computed using the quasi-local energy-momentum tensor of a gravitational bulk action S_{gr} ,

$$T^{ab} = \frac{2}{\sqrt{-\det\gamma}} \frac{\delta S_{\text{gr}}}{\delta \gamma_{ab}}. \quad (3.2)$$

The action S_{gr} which is defined on an asymptotically AdS space-time M is viewed as functional of the boundary metric γ_{ab} on ∂M . The resulting T^{ab} typically diverge, but it is always possible to obtain finite results by adding an appropriately chosen boundary counter-term whose form depends on the dimensionality of space-time. Holographic renormalization provides a well defined prescription for implementing the Brown-York procedure, [22], without using a reference space-time to subtract the infinities.

In AdS_4/CFT_3 correspondence, in particular, the gravitational action consists of bulk and boundary terms chosen as follows, [20],

$$S_{\text{gr}} = -\frac{1}{2\kappa^2} \int_M d^4x \sqrt{-\det g} (R[g] + 2\Lambda) - \frac{1}{\kappa^2} \int_{\partial M} d^3x \sqrt{-\det\gamma} K - \frac{2}{\kappa^2} \sqrt{-\frac{\Lambda}{3}} \int_{\partial M} d^3x \sqrt{-\det\gamma} \left(1 + \frac{3}{4\Lambda} R[\gamma]\right). \quad (3.3)$$

The first boundary contribution is the usual Gibbons-Hawking term written in terms of the trace of the second fundamental form, i.e., the extrinsic mean curvature

$$K = \gamma^{ab} K_{ab}, \quad (3.4)$$

associated to the embedding of ∂M in M . The second boundary contribution is the contact term needed to remove all divergencies in the present case.

Then, according to definition, the energy-momentum tensor of the field theory is expressed in terms of the intrinsic and extrinsic geometry of the AdS boundary at infinity, as

$$\kappa^2 T_{ab} = K_{ab} - K \gamma_{ab} - 2 \sqrt{-\frac{\Lambda}{3}} \gamma_{ab} + \sqrt{-\frac{3}{\Lambda}} G_{ab}. \quad (3.5)$$

Here, G_{ab} denotes the Einstein tensor of the induced three-dimensional metric γ_{ab} ,

$$G_{ab} = R_{ab}[\gamma] - \frac{1}{2} R[\gamma] \gamma_{ab}. \quad (3.6)$$

Clearly, only boundary terms contribute to the answer since the bulk metric is always taken to satisfy the classical gravitational equations of motion.

In practice, the computation is performed by first writing the metric g on M in the form

$$ds^2 = N^2 dr^2 + \gamma_{ab} (dx^a + N^a dr) (dx^b + N^b dr) \quad (3.7)$$

using appropriately chosen (N, N^a) functions, as in an ADM-like decomposition. The three-dimensional surface arising at fixed distance r serves as boundary ∂M_r to the interior four-dimensional region M_r . The induced metric on ∂M_r is γ_{ab} evaluated at the boundary value of r , which is held finite at this point. A useful relation among the bulk and boundary metrics is

$$\sqrt{-\det g} = N \sqrt{-\det\gamma}. \quad (3.8)$$

The second fundamental form K_{ab} on ∂M_r is defined using the outward pointing normal vector η_μ to the boundary ∂M_r with components

$$\eta_\mu = N \delta_\mu^r. \tag{3.9}$$

In particular, one has

$$K_{ab} = -\nabla_{(a}\eta_{b)}, \tag{3.10}$$

using the covariant derivatives with respect to the bulk metric g ; in the present case the expressions simplify to

$$K_{ab} = N \Gamma_{ab}^r[g]. \tag{3.11}$$

At the end of the computation, T^{ab} on the AdS boundary ∂M is obtained by letting $r \rightarrow \infty$.

Since the boundary metric acquires an infinite Weyl factor as r is taken to infinity, it is more appropriate to think of the AdS boundary as a conformal class of boundaries and define \mathcal{S} as the boundary space-time with metric

$$ds_{\mathcal{S}}^2 = \lim_{r \rightarrow \infty} \left(-\frac{3}{\Lambda r^2} \gamma_{ab} dx^a dx^b \right). \tag{3.12}$$

Then, the renormalized energy-momentum tensor on \mathcal{S} is defined accordingly by

$$T_{ab}^{\text{renorm}} = \lim_{r \rightarrow \infty} \left(\sqrt{-\frac{\Lambda}{3}} r T_{ab} \right) \tag{3.13}$$

and it is finite. This is the quantity that we will compute for all different type of gravitational perturbations of AdS_4 black holes.

As for the trace of the energy-momentum tensor on the three-dimensional boundary ∂M_r ,

$$\kappa^2 T^a{}_a = -2K - \frac{1}{2} \sqrt{-\frac{3}{\Lambda}} R[\gamma] - 6 \sqrt{-\frac{\Lambda}{3}}, \tag{3.14}$$

it has the following leading behavior for large r ,

$$T^a{}_a \sim \frac{1}{r^4}. \tag{3.15}$$

Terms of order $1/r^3$ are vanishing in this case by the absence of conformal anomalies in three dimensions, [17], and, therefore, the trace of the renormalized energy-momentum tensor vanishes.

3.2 Static AdS_4 black holes

We first apply the formalism to the simple example of static AdS_4 Schwarzschild solution that will be subsequently used as reference frame to study the effect of linear perturbations. All steps of the calculation are included for illustrative reasons.

In this case, we have the following choice of (N, N^a) functions,

$$N = \frac{1}{\sqrt{f(r)}}, \quad N^a = 0, \tag{3.16}$$

and the induced metric on ∂M_r is

$$\gamma_{ab} = \begin{pmatrix} -f(r) & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}. \quad (3.17)$$

The second fundamental form turns out to be

$$K_{ab} = \sqrt{f(r)} \begin{pmatrix} f'(r)/2 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & -r \sin^2 \theta \end{pmatrix} \quad (3.18)$$

and its trace is

$$K = -\frac{1}{2r\sqrt{f(r)}} (rf'(r) + 4f(r)). \quad (3.19)$$

Also, the Ricci curvature tensor of the induced metric γ takes the simple form

$$R_{ab}[\gamma] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sin^2 \theta \end{pmatrix} \quad (3.20)$$

and the Ricci scalar curvature is

$$R[\gamma] = \frac{2}{r^2}. \quad (3.21)$$

Then, following the general prescription for computing the energy-momentum tensor in AdS gravity, we find the following expressions on ∂M_r ,

$$\kappa^2 T_{tt} = \frac{f(r)}{r^2} \left(\sqrt{-\frac{3}{\Lambda}} + 2r^2 \sqrt{-\frac{\Lambda}{3}} - 2r\sqrt{f(r)} \right), \quad (3.22)$$

$$\kappa^2 T_{\theta\theta} = \frac{r}{\sqrt{f(r)}} \left(f(r) + \frac{r}{2} f'(r) \right) - 2r^2 \sqrt{-\frac{\Lambda}{3}} \quad (3.23)$$

and $T_{\phi\phi} = \sin^2 \theta T_{\theta\theta}$, whereas all other components are zero.

As $r \rightarrow \infty$, ∂M_r is pushed away to spatial infinity and the energy-momentum tensor admits the following asymptotic expansion

$$\kappa^2 T_{tt} = \frac{2m}{r} \sqrt{-\frac{\Lambda}{3}} + \frac{1}{4r^2} \sqrt{-\frac{3}{\Lambda}} + \mathcal{O}\left(\frac{1}{r^3}\right), \quad (3.24)$$

$$\kappa^2 T_{\theta\theta} = \frac{m}{r} \sqrt{-\frac{3}{\Lambda}} + \frac{1}{4r^2} \left(\sqrt{-\frac{3}{\Lambda}} \right)^3 + \mathcal{O}\left(\frac{1}{r^3}\right), \quad (3.25)$$

whereas $T_{\phi\phi} = \sin^2\theta T_{\theta\theta}$, as before. Note at this point that the trace of the energy-momentum tensor is

$$\kappa^2 T^a{}_a = \frac{1}{4r^4} \left(\sqrt{-\frac{3}{\Lambda}} \right)^3 + \mathcal{O} \left(\frac{1}{r^5} \right) \quad (3.26)$$

exhibiting the correct asymptotic behavior due to the absence of conformal anomaly.

The three-dimensional metric on \mathcal{S} is the Lorentzian conformally flat metric on $R \times S^2$ written in spherical coordinates,

$$ds_{\mathcal{S}}^2 = -dt^2 - \frac{3}{\Lambda} (d\theta^2 + \sin^2\theta d\phi^2) . \quad (3.27)$$

The renormalized energy-momentum tensor of the boundary theory has the following non-vanishing components

$$\kappa^2 T_{tt}^{(0)} = -\frac{2m\Lambda}{3}, \quad \kappa^2 T_{\theta\theta}^{(0)} = m, \quad \kappa^2 T_{\phi\phi}^{(0)} = m \sin^2\theta, \quad (3.28)$$

reproducing the expressions already known in the literature. The superscript (0) is used for reference to the static background.

3.3 Axial perturbations

Axial perturbations of AdS Schwarzschild black-holes are parametrized by two radial functions $h_0(r)$ and $h_1(r)$. The four-dimensional metric has coefficients

$$N = \frac{1}{\sqrt{f(r)}}, \quad N_{\phi} = h_1(r) e^{-i\omega t} \sin\theta \partial_{\theta} P_l(\cos\theta), \quad N_t = 0 = N_{\theta} \quad (3.29)$$

and the induced three-dimensional metric on ∂M_r is a perturbation of the static metric

$$\gamma_{ab} = \gamma_{ab}^{(0)} + \begin{pmatrix} 0 & 0 & h_0(r) \\ 0 & 0 & 0 \\ h_0(r) & 0 & 0 \end{pmatrix} e^{-i\omega t} \sin\theta \partial_{\theta} P_l(\cos\theta) . \quad (3.30)$$

The second fundamental form is also a perturbation of the second fundamental form of the static solution,

$$K_{ab} = K_{ab}^{(0)} + \delta K_{ab} \quad (3.31)$$

and the same thing applies to the Ricci curvature tensor of the metric γ_{ab} ,

$$R_{ab}[\gamma] = R_{ab}^{(0)} + \delta R_{ab} . \quad (3.32)$$

However, the traces of K_{ab} and R_{ab} are inert to the perturbations, i.e.,

$$K = K^{(0)}, \quad R[\gamma] = R^{(0)}, \quad (3.33)$$

which in turn imply that the trace of the boundary energy-momentum tensor coincides with the result obtained earlier for the static background,

$$T^a{}_a = T^{(0)a}{}_a . \quad (3.34)$$

This can be regarded as consistency check for the cancelation of conformal anomalies for axial perturbations of the metric.

The complete energy-momentum tensor of the boundary theory on ∂M_r assumes the following form,

$$T_{ab} = T_{ab}^{(0)} + \begin{pmatrix} 0 & 0 & \delta T_{t\phi} \\ 0 & 0 & \delta T_{\theta\phi} \\ \delta T_{t\phi} & \delta T_{\theta\phi} & 0 \end{pmatrix}, \quad (3.35)$$

where $\delta T_{t\phi}$ and $\delta T_{\theta\phi}$ are given explicitly in appendix B.

Using the asymptotic expansion of the metric functions $h_0(r)$ and $h_1(r)$ at spatial infinity, as given in appendix A, we find that all divergencies of δT_{ab} cancel as $r \rightarrow \infty$ irrespective of boundary conditions. In particular, after conformal rescaling, the three-dimensional metric on \mathcal{I} takes the form

$$ds_{\mathcal{I}}^2 = -dt^2 - \frac{3}{\Lambda} (d\theta^2 + \sin^2\theta d\phi^2) + 2\frac{iI_0}{\omega} e^{-i\omega t} \sin\theta \partial_{\theta} P_l(\cos\theta) dt d\phi \quad (3.36)$$

and the non-vanishing components of the axial perturbations of the renormalized energy-momentum tensor are

$$\kappa^2 \delta T_{t\phi} = -\frac{i\Lambda}{6\omega} \left(2mI_0 + (l-1)(l+2) \left(\frac{3i\omega}{\Lambda} I_0 - I_1 \right) \right) e^{-i\omega t} \sin\theta \partial_{\theta} P_l(\cos\theta), \quad (3.37)$$

$$\kappa^2 \delta T_{\theta\phi} = \frac{1}{2} \left(\frac{3i\omega}{\Lambda} I_0 - I_1 \right) e^{-i\omega t} \sin\theta [l(l+1) P_l(\cos\theta) + 2\cot\theta \partial_{\theta} P_l(\cos\theta)]. \quad (3.38)$$

It can be verified independently, as consistency check, that the total energy-momentum tensor is traceless and conserved on \mathcal{I} .

3.4 Polar perturbations

Polar perturbations of AdS Schwarzschild black-holes are parametrized by three radial functions $H_0(r)$, $H_1(r)$ and $K(r)$. In this case, the four-dimensional metric admits an ADM-like decomposition with coefficients

$$N = \frac{1}{\sqrt{f(r)}} \left(1 + \frac{1}{2} H_0(r) e^{-i\omega t} P_l(\cos\theta) \right) \quad (3.39)$$

and

$$N_t = H_1(r) e^{-i\omega t} P_l(\cos\theta), \quad N_{\theta} = 0 = N_{\phi}. \quad (3.40)$$

Also, the induced three-dimensional metric on ∂M_r is a perturbation of the static metric with diagonal form

$$\gamma_{ab} = \gamma_{ab}^{(0)} + \begin{pmatrix} f(r)H_0(r) & 0 & 0 \\ 0 & r^2K(r) & 0 \\ 0 & 0 & r^2K(r)\sin^2\theta \end{pmatrix} e^{-i\omega t} P_l(\cos\theta). \quad (3.41)$$

The second fundamental form is a perturbation of the corresponding static expression, as before, and the same thing applies to the Ricci curvature tensor of the corresponding metric γ_{ab} . It turns out that the trace of the second fundamental form for polar perturbations is

$$K = K^{(0)} + \left(\frac{1}{4\sqrt{f(r)}} (f'(r)H_0(r) + 2f(r)H_0'(r) + 4i\omega H_1(r)) + \frac{1}{r}\sqrt{f(r)} (H_0(r) - rK'(r)) \right) e^{-i\omega t} P_l(\cos\theta) \quad (3.42)$$

and the Ricci curvature scalar is

$$R[\gamma] = R^{(0)} - \frac{1}{r^2} \left(\left(2\frac{\omega^2 r^2}{f(r)} - (l-1)(l+2) \right) K(r) + l(l+1)H_0(r) \right) e^{-i\omega t} P_l(\cos\theta). \quad (3.43)$$

Note that $\delta R_{\phi\phi} \neq \sin^2\theta \delta R_{\theta\theta}$, which will in turn imply that $\delta T_{\phi\phi} \neq \sin^2\theta \delta T_{\theta\theta}$ for the corresponding components of the energy-momentum tensor. It follows that the trace of the boundary energy-momentum on ∂M_r is not inert to these perturbations, since

$$\begin{aligned} \kappa^2 T^a{}_a = \kappa^2 T^{(0)a}{}_a - & \left(\left(\frac{2}{r}\sqrt{f(r)} + \frac{f'(r)}{2\sqrt{f(r)}} - \frac{l(l+1)}{2r^2}\sqrt{-\frac{3}{\Lambda}} \right) H_0(r) \right. \\ & + \sqrt{f(r)} H_0'(r) - \frac{1}{2r^2}\sqrt{-\frac{3}{\Lambda}} \left(2\frac{\omega^2 r^2}{f(r)} - (l-1)(l+2) \right) K(r) \\ & \left. - 2\sqrt{f(r)} K'(r) + \frac{2i\omega}{\sqrt{f(r)}} H_1(r) \right) e^{-i\omega t} P_l(\cos\theta). \end{aligned} \quad (3.44)$$

However, as we will see shortly, the additional terms are of order $\mathcal{O}(1/r^4)$ when $r \rightarrow \infty$, in agreement with the cancelation of conformal anomalies.

The complete energy-momentum tensor of the boundary theory on ∂M_r takes a form that is complementary to the corresponding expression for axial perturbations, namely

$$T_{ab} = T_{ab}^{(0)} + \begin{pmatrix} \delta T_{tt} & \delta T_{t\theta} & 0 \\ \delta T_{t\theta} & \delta T_{\theta\theta} & 0 \\ 0 & 0 & \delta T_{\phi\phi} \end{pmatrix}, \quad (3.45)$$

where the corresponding expressions are given explicitly in appendix B.

Using the asymptotic expansion of the metric functions $H_0(r)$, $H_1(r)$ and $K(r)$ at spatial infinity, as given in appendix A, we find that all divergencies cancel as $r \rightarrow \infty$ irrespective of boundary conditions and all works well as required on general grounds. In this case, the three-dimensional metric on the boundary takes the following form, after conformal rescaling,

$$ds_{\mathcal{J}}^2 = -dt^2 - \frac{3}{\Lambda} [1 + Re^{-i\omega t} P_l(\cos\theta)] (d\theta^2 + \sin^2\theta d\phi^2), \quad (3.46)$$

where $R = K(r = \infty)$ is the following function of ω

$$R = \frac{\Lambda}{3} J_1 - \left(i\omega - \frac{2m\Lambda}{(l-1)(l+2)} \right) J_0 . \quad (3.47)$$

Explicit calculation shows that the non-vanishing components of the polar perturbations of the renormalized energy-momentum tensor are:

$$\kappa^2 \delta T_{tt} = m\Lambda(R - i\omega_s J_0) e^{-i\omega t} P_l(\cos\theta) , \quad (3.48)$$

$$\begin{aligned} \kappa^2 \delta T_{\theta\theta} = & \frac{1}{4} \left(4m \frac{l(l+1)+1}{(l-1)(l+2)} R - l(l+1) \left(1 + \frac{3\omega^2}{\Lambda} \right) J_0 \right) e^{-i\omega t} P_l(\cos\theta) + \\ & + \frac{1}{4} \left(\frac{12mR}{(l-1)(l+2)} - \left(l(l+1) + \frac{6\omega^2}{\Lambda} \right) J_0 \right) e^{-i\omega t} \cot\theta \partial_\theta P_l(\cos\theta) , \end{aligned} \quad (3.49)$$

$$\begin{aligned} \kappa^2 \delta T_{\phi\phi} = & -\frac{1}{4} \left(4m \frac{2l(l+1)-1}{(l-1)(l+2)} R - l(l+1) \left(l(l+1) - 1 + \frac{3\omega^2}{\Lambda} \right) J_0 \right) e^{-i\omega t} \sin^2\theta P_l(\cos\theta) - \\ & - \frac{1}{4} \left(\frac{12mR}{(l-1)(l+2)} - \left(l(l+1) + \frac{6\omega^2}{\Lambda} \right) J_0 \right) e^{-i\omega t} \sin\theta \cos\theta \partial_\theta P_l(\cos\theta) , \end{aligned} \quad (3.50)$$

$$\kappa^2 \delta T_{t\theta} = \frac{1}{4} i\omega(l-1)(l+2) J_0 e^{-i\omega t} \partial_\theta P_l(\cos\theta) . \quad (3.51)$$

It can be verified, as consistency check, that the complete energy-momentum tensor is traceless and conserved on \mathcal{S} .

Note that the renormalized δT_{tt} vanishes only when $R = i\omega_s J_0$. These are mixed boundary conditions for the polar perturbations that are supersymmetric partner to perfectly reflecting boundary conditions, $I_0 = 0$, for the axial perturbations.

4. Hydrodynamic representation

The energy-momentum tensor associated to the static AdS_4 black hole represents a perfect conformal fluid on the three-dimensional boundary with metric $g_{ab}^{(0)}$, velocity vector $u_a = (-1, 0, 0)$ and energy density

$$\kappa^2 \rho = -\frac{2m\Lambda}{3} . \quad (4.1)$$

Thus, it makes sense to compare the fluctuations of the energy-momentum tensor for linear perturbations of black holes with the theory of first order hydrodynamics. The comparison is only formal in the general case, but the representation of the results for the energy-momentum tensor in terms of fluid dynamics will be helpful in the sequel. The true hydrodynamic modes of black hole physics will also be discussed in this section.

4.1 First order hydrodynamics

Recall that the energy momentum tensor of a perfect relativistic fluid takes the following form

$$T_{ab} = \rho u_a u_b + p \Delta_{ab} , \quad (4.2)$$

where

$$\Delta_{ab} = u_a u_b + g_{ab} \quad (4.3)$$

is given in terms of the unit velocity vector $u_a u^a = -1$ and the metric. Conformal fluids have energy-momentum tensor with zero trace and therefore $\rho = 2p$ in three dimensions.

Deviations from the perfect fluid form are parametrized by adding appropriate viscosity terms. Since the hydrodynamic velocity is ambiguous for non-equilibrium processes one should make a (physically insignificant) choice. We will use the so called *energy frame*, meaning that u_a is the unit time-like eigenvector of T_{ab} defined as

$$T_{ab} u^b = -\rho u_a \tag{4.4}$$

Then, the energy-momentum tensor of a general relativistic fluid admits the following decomposition (see, for instance, the textbook [26]),

$$T_{ab} = \rho u_a u_b + p \Delta_{ab} + \Pi_{ab}, \tag{4.5}$$

where ρ , p are the corresponding energy density and pressure fields. Π^{ab} is a transverse tensor, $u_a \Pi^{ab} = 0$, that describes the viscous part of the energy-momentum tensor of the fluid, and, in general, it admits an expansion in the derivatives of u^a ,

$$\Pi_{ab} = \Pi_{ab}^{(1)} + \Pi_{ab}^{(2)} + \dots \tag{4.6}$$

First order hydrodynamics is concerned with the structure of $\Pi_{(1)}^{ab}$ and is well studied. In this case, using the energy frame, we have, [26],

$$\Pi_{(1)}^{ab} = -\eta \sigma^{ab} - \zeta \Delta^{ab} (\nabla_c u^c), \tag{4.7}$$

where

$$\sigma^{ab} = 2\nabla^{<a} u^{b>} \tag{4.8}$$

expresses the symmetric, transverse and traceless part of Π^{ab} up to first derivatives in u^a . Here, we use the notation (adapted to three-dimensional fluids) of the bracketed second rank tensor

$$A^{<ab>} = \frac{1}{2} \left(\Delta^{ac} \Delta^{bd} (A_{cd} + A_{dc}) - \Delta^{ab} \Delta^{cd} A_{cd} \right), \tag{4.9}$$

which is transverse, $u_a A^{<ab>} = 0$, and traceless, $g_{ab} A^{<ab>} = 0$. The coefficients η and ζ depend in general on ρ and they are called shear and bulk viscosity, respectively. Of course, conformal fluids have $\zeta = 0$, whereas the value of η depends on the particular case.

In this context, one may also consider the vorticity of the velocity vector field u_a , which is defined as follows,

$$\Omega^{ab} = \frac{1}{2} \Delta^{ac} \Delta^{bd} (\nabla_c u_d - \nabla_d u_c), \tag{4.10}$$

and it is clearly antisymmetric. As will be seen shortly, axial and polar perturbations can be distinguished from each other by their vorticity tensor field.

4.2 Formal identifications

Applying first order hydrodynamics to the perturbations of AdS_4 black holes we arrive at the following formal identifications regarding the shear viscosity coefficient:

(i) Axial perturbations: using the energy-momentum tensor computed for axial perturbations with general boundary conditions and the associated metric on \mathcal{I} , one easily finds that the normalized time-like unit vector u_a has components

$$u_t = -1, \quad u_\theta = 0 \quad (4.11)$$

and

$$u_\phi = -\frac{i}{6m\omega}(l-1)(l+2) \left(\frac{3i\omega}{\Lambda} I_0 - I_1 \right) e^{-i\omega t} \sin\theta \partial_\theta P_l(\cos\theta) \quad (4.12)$$

within the linear approximation. Also, the corresponding energy density is

$$\kappa^2 \rho = -\frac{2m\Lambda}{3}, \quad (4.13)$$

as in the unperturbed black hole case.

Explicit computation shows that all components of $\Pi_{ab}^{(1)}$ vanish within the linear approximation apart from

$$\kappa^2 \Pi_{\theta\phi}^{(1)} = \frac{1}{2} \left(\frac{3i\omega}{\Lambda} I_0 - I_1 \right) e^{-i\omega t} \sin\theta [l(l+1)P_l(\cos\theta) + 2\cot\theta \partial_\theta P_l(\cos\theta)]. \quad (4.14)$$

Likewise, the computation of σ_{ab} also shows that all its components vanish apart from $\sigma_{\theta\phi}$. The result turns out to be identical to $\Pi_{\theta\phi}$ up to an overall factor that determines the coefficient η of shear viscosity for axial perturbation. Direct comparison, within the context of first order hydrodynamics, yields

$$\kappa^2 \eta = \frac{3im\omega S}{(l-1)(l+2)(I_0 + S)}, \quad (4.15)$$

where it is set for convenience

$$S = \frac{(l-1)(l+2)}{6m} \left(\frac{3i\omega}{\Lambda} I_0 - I_1 \right). \quad (4.16)$$

As special case, we refer to axial perturbations satisfying perfectly reflecting boundary conditions, $I_0 = 0$, for which it turns out that

$$\kappa^2 \eta = \frac{3im\omega}{(l-1)(l+2)}. \quad (4.17)$$

The axial perturbations have a vorticity field with non-vanishing component

$$\Omega^{\theta\phi} = -\frac{\Lambda^2 \omega_s}{9\omega} \left(\frac{3i\omega}{\Lambda} I_0 - I_1 \right) e^{-i\omega t} \sin\theta P_l(\cos\theta) \quad (4.18)$$

under general boundary conditions.

(ii) **Polar perturbations:** similar considerations for polar perturbations satisfying general boundary conditions yield the normalized time-like unit vector with components

$$u_t = -1, \quad u_\phi = 0, \quad (4.19)$$

and

$$u_\theta = \frac{i\omega}{4m\Lambda}(l-1)(l+2)J_0 e^{-i\omega t} \partial_\theta P_l(\cos\theta), \quad (4.20)$$

whereas the corresponding energy density turns out to be

$$\kappa^2 \rho = -\frac{2m\Lambda}{3} + m\Lambda(R - i\omega_s J_0) e^{-i\omega t} P_l(\cos\theta). \quad (4.21)$$

Explicit computation of the tensor $\Pi_{ab}^{(1)}$ yields

$$\kappa^2 \Pi_{\theta\theta}^{(1)} = \frac{1}{8} \left(\frac{12mR}{(l-1)(l+2)} - \left(l(l+1) + \frac{6\omega^2}{\Lambda} \right) J_0 \right) e^{-i\omega t} \times [l(l+1)P_l(\cos\theta) + 2\cot\theta \partial_\theta P_l(\cos\theta)] \quad (4.22)$$

and

$$\Pi_{\phi\phi}^{(1)} = -\sin^2\theta \Pi_{\theta\theta}^{(1)}, \quad (4.23)$$

in agreement with its traceless property. All other components of Π_{ab} vanish within the linear approximation. To compare with first order hydrodynamics we also compute σ_{ab} and find that its components vanish apart from $\sigma_{\theta\theta}$ and $\sigma_{\phi\phi}$. Their expressions are proportional to $\Pi_{\theta\theta}$ and $\Pi_{\phi\phi}$, respectively, and comparison yields the following coefficient η of shear viscosity for polar perturbations,

$$\kappa^2 \eta = -\frac{im\Lambda}{2\omega(l-1)(l+2)} \left(\frac{12mR}{(l-1)(l+2)J_0} - \left(l(l+1) + \frac{6\omega^2}{\Lambda} \right) \right). \quad (4.24)$$

The special case of polar perturbations with mixed boundary conditions $R = i\omega_s J_0$, which are supersymmetric partner to perfectly reflecting boundary conditions on the axial perturbations, leads to the coefficient

$$\kappa^2 \eta = \frac{3im\omega}{(l-1)(l+2)}. \quad (4.25)$$

This value is identical to the shear viscosity of axial perturbations with perfectly reflecting boundary conditions.

The polar perturbations always have vanishing vorticity, which distinguishes them from the axial perturbation.

4.3 True hydrodynamic modes

The hydrodynamic representation of the energy-momentum tensor of black hole perturbations is just a convenient (yet formal) way to rewrite the results of the calculation. Nevertheless, there is a fundamental relation between the physics of black holes and relativistic hydrodynamics that goes beyond first order and extends to higher order causal theories

of fluid dynamics, [27–31]. The hydrodynamic equations can be thought as an effective theory describing the dynamics of the system at large length and time scales. The true hydrodynamic modes of black hole perturbations are identified by computing the retarded two-point Green functions of the energy momentum tensor and finding their behavior at zero spatial momentum for low frequencies (for an overview, see, for instance, [24], and references therein). A rather general result has emerged in this context in recent years, namely that the ratio of shear viscosity to the entropy density of a very large AdS black hole assumes a universal value, [23]. More precisely, it turns out that the true hydrodynamic modes have shear viscosity

$$\kappa^2 \eta = \frac{m}{r_h} \tag{4.26}$$

that is independent of l , and, therefore, the ratio of shear viscosity to entropy density is

$$\frac{\eta}{s} = \frac{4}{r_h^2} \left(-\frac{3}{\Lambda} \right) \eta = \frac{1}{4\pi} \tag{4.27}$$

in units where Boltzmann’s constant and Planck’s constant are set equal to 1. This result appears to be valid in all dimensions and it has been further argued that it provides an absolute lower bound (known as KSS bound) for the ratio η/s of all substances in nature; see also the general presentations [24] by the same authors.

The first example of true hydrodynamic modes is provided by purely dissipative modes with frequencies

$$\Omega_s = -i \frac{(l-1)(l+2)}{3r_h}, \tag{4.28}$$

which turn out to belong to the spectrum of axial perturbations satisfying Dirichlet boundary conditions, $I_0 = 0$, up to $\mathcal{O}(1/r_h^2)$ corrections, [8, 32]. Thus, for very large AdS_4 black holes the values Ω_s are exact and the corresponding shear viscosity coefficient, as calculated earlier, is

$$\kappa^2 \eta = \frac{3im\omega}{(l-1)(l+2)} = \frac{m}{r_h} \tag{4.29}$$

and yields the KSS value. Polar perturbations with mixed boundary conditions $R = i\omega_s$ also admit purely dissipative modes with frequencies Ω_s and yield the same result (4.29) for very large black holes.

Another example of true hydrodynamic modes is provided by the complex values of frequency

$$\Omega_{\pm} = \pm \sqrt{-\frac{\Lambda}{6}l(l+1)} - i \frac{(l-1)(l+2)}{6r_h}, \tag{4.30}$$

which turn out to belong to the spectrum of polar perturbations satisfying mixed boundary conditions $R = 0$, up to $\mathcal{O}(1/r_h^2)$ corrections, [32]. Thus, for very large AdS_4 black holes the values Ω_{\pm} are exact and the corresponding shear viscosity coefficient, as calculated earlier, turns out to be

$$\kappa^2 \eta = \frac{im\Lambda}{2\omega(l-1)(l+2)} \left(l(l+1) + \frac{6\omega^2}{\Lambda} \right) = \frac{m}{r_h}, \tag{4.31}$$

up to $\mathcal{O}(1/r_h^2)$ corrections, and it yields the KSS value, as before. Axial perturbations with mixed boundary conditions

$$\frac{I_1}{I_0} = \frac{3}{\Lambda} i\omega \left(1 - \frac{\omega}{\omega_s}\right) + \frac{6m}{(l-1)(l+2)} \tag{4.32}$$

are supersymmetric partner to polar perturbations with $R = 0$ and as such they also admit quasi-normal modes with complex frequencies Ω_{\pm} . Comparison with the corresponding shear viscosity coefficient yields the same result (4.31) for very large black holes.

Actually, one can easily show that the only supersymmetric partner boundary conditions that yield

$$\eta_{\text{axial}} = \eta_{\text{polar}} , \tag{4.33}$$

as computed explicitly in the previous subsection on general grounds, are (i) $I_0 = 0$ and $R = i\omega_s J_0$, and (ii) I_1/I_0 given by equation (4.32) and $R = 0$; all other boundary conditions yield $\eta_{\text{axial}} \neq \eta_{\text{polar}}$. Furthermore, by demanding

$$\eta_{\text{axial}} = \eta_{\text{polar}} = m/r_h , \tag{4.34}$$

it follows from the analysis above that the only allowed frequencies are Ω_s and Ω_{\pm} when $r_h \rightarrow \infty$.

Gravitational perturbations associated to true hydrodynamic modes (of either type) satisfying the above special boundary conditions will be particularly relevant in section 6.

5. Connection with the normalized Ricci flow on S^2

The observation made in the literature, as result of numerical investigations, that very large AdS_4 black holes exhibit purely dissipative modes for axial perturbations satisfying perfectly reflecting Dirichlet boundary conditions with frequencies (4.28), Ω_s , calls for an analytic explanation. The same set of modes also arise for polar perturbations satisfying mixed boundary conditions that are supersymmetric partner to the axial perturbations of very large AdS_4 black holes with Dirichlet boundary conditions.

Recall at this point that there is a second order geometric evolution equation for metrics on a Riemannian manifold driven by the Ricci curvature tensor,

$$\partial_u g_{\mu\nu} = -R_{\mu\nu} , \tag{5.1}$$

known as Ricci flow (see, for instance, the collection of selected works [33]). The volume of space is not preserved under the evolution, but it is always possible to define a variant, known as normalized Ricci flow, which is volume preserving. The Ricci flow for the class of conformally flat metrics on S^2 ,

$$ds_2^2 = 2e^{\Phi(z, \bar{z}; u)} dz d\bar{z} , \tag{5.2}$$

takes the following form

$$\partial_u \Phi = e^{-\Phi} \partial \bar{\partial} \Phi , \tag{5.3}$$

whereas the corresponding normalized Ricci flow on S^2 with fixed area 4π is given by

$$\partial_u \Phi = e^{-\Phi} \partial \bar{\partial} \Phi + 1 . \tag{5.4}$$

The constant curvature metric provides the fixed point for the normalized Ricci flow equation on S^2 . In fact, the canonical metric is reached from any given initial data after sufficiently long time. It is instructive to examine the spectrum of linear perturbations around this equilibrium state at late times, using small axially symmetric deformations of the round unit sphere parametrized by $\epsilon_l(u) P_l(\cos\theta)$,

$$ds_2^2 = [1 + \epsilon_l(u) P_l(\cos\theta)] (d\theta^2 + \sin^2\theta d\phi^2) . \tag{5.5}$$

It can be easily verified that the normalized Ricci flow yields the following characteristic decay of metric perturbations, as $u \rightarrow \infty$,

$$\epsilon_l(u) = \epsilon_l(0) \exp\left(-\frac{u}{2}(l-1)(l+2)\right) . \tag{5.6}$$

Then, the spectrum of purely imaginary frequencies associated to the normalized Ricci flow is given by

$$\Omega_s \sim -i \frac{(l-1)(l+2)}{2} , \tag{5.7}$$

up to a universal factor that depends on the physical scale of u and can be identified with $3r_h/2$ to match the values (4.28).

In view of this relation, it is natural to expect that there is an embedding of the (normalized) Ricci flow into Einstein equations so that the resulting four-dimensional metric describes a new radiative class of space-times. In this context, u should have the interpretation of retarded time and the AdS_4 black hole should arise as a fixed point (static) configuration after all radiation has been damped away. Also, in this context, $\tau_s = 1/i\Omega_s$ should be the characteristic time scale, depending on l , for the multi-pole gravitational radiation damping close to equilibrium. We do not expect this embedding to exist when $\Lambda = 0$ nor to be exact in the non-linear regime when the size of the black hole is not very large. This idea might be more natural to implement in the polar sector which resembles the perturbations (5.5) in the spherical part of the four-dimensional metric.

It will also be interesting to have an analogous analytic explanation for the existence of the complex frequencies Ω_{\pm} in the quasi-normal mode spectrum of very large AdS_4 black holes. The boundary conditions are different in this case and, therefore, the geometric framework that may account for their presence will not be the same.

6. Energy-momentum/Cotton tensor duality

In this section we describe the main application of the results for the energy-momentum tensor of perturbed black holes. We first introduce the notion of Cotton tensor in three dimensions, using the Chern-Simons gravitational action, and then compare the two expressions for suitably selected boundary conditions.

6.1 Chern-Simons gravitational action

In three dimensions there is a quantity that remains invariant under local conformal changes of the metric γ_{ab} and vanishes if and only if the metric is conformally flat. It is provided by the density $\sqrt{-\det\gamma} C^a_b$, where C_{ab} is an odd parity tensor, called Cotton tensor,

$$\begin{aligned} C^{ab} &= \frac{1}{2\sqrt{-\det\gamma}} \left(\epsilon^{acd} \nabla_c R^b_d + \epsilon^{bcd} \nabla_c R^a_d \right) \\ &= \frac{\epsilon^{acd}}{\sqrt{-\det\gamma}} \nabla_c \left(R^b_d - \frac{1}{4} \delta^b_d R \right) \end{aligned} \tag{6.1}$$

with $\epsilon^{t\theta\phi} = 1$. The Cotton tensor is symmetric, traceless and identically covariantly conserved. As such, it arises as functional derivative of a geometric invariant, namely the three-dimensional gravitational Chern-Simons action, [34],

$$C_{ab} = \frac{1}{\sqrt{-\det\gamma}} \frac{\delta S_{\text{CS}}}{\delta \gamma^{ab}}, \tag{6.2}$$

where

$$S_{\text{CS}} = \frac{1}{2} \int d^3x \sqrt{-\det\gamma} \epsilon^{abc} \Gamma^d_{ae} \left(\partial_b \Gamma^e_{cd} + \frac{2}{3} \Gamma^e_{bf} \Gamma^f_{cd} \right). \tag{6.3}$$

S_{CS} is an action of third order in the dynamical variables of the theory.

The gravitational Chern-Simons action on the boundary of asymptotically locally AdS_4 backgrounds arises from the topological Hirzebruch-Pontryagin action on the bulk space-time, namely

$$\int d^4x \sqrt{-\det g} R_{abcd} \star R^{abcd} = \frac{1}{2} \int d^4x \epsilon^{abef} R_{abcd} R_{ef}{}^{cd}, \tag{6.4}$$

since the integrand is a total derivative; the $R \wedge R$ action enters into the definition of the signature $\tau(M)$. When the perturbations of black holes satisfy general boundary conditions, so that the boundary metric is not conformally flat, the corresponding Cotton tensor is non-vanishing. Thus, adding S_{CS} to the boundary action improves the boundary energy-momentum tensor by the Cotton tensor and changes the characteristics of the fluid velocity field in the hydrodynamic representation of the problem; for example, the polar sector, which has no vorticity, acquires some by this modification. We will not pursue this general connection further in the present exposition. Instead, we will restrict ourselves to the rather curious observation that the Cotton tensor and the energy-momentum tensor of black hole perturbations exhibit an *axial-polar duality* with respect to appropriately chosen supersymmetric partner boundary conditions.

6.2 The new correspondence for black holes

We are now in position to establish the relation between the energy-momentum tensor of black hole perturbations and the Cotton tensor of a dual boundary metric by studying separately the polar and axial cases.

(i) Polar perturbations: let us first consider the boundary metric for polar perturbations of AdS_4 black holes, which is given in general by

$$ds_{\mathcal{J}}^2(\text{polar}) = -dt^2 - \frac{3}{\Lambda}[1 + Re^{-i\omega t}P_l(\cos\theta)](d\theta^2 + \sin^2\theta d\phi^2). \quad (6.5)$$

Straightforward computation shows that its Cotton tensor has the following non-vanishing components,

$$C_{\theta\phi} = \frac{i\omega}{4}Re^{-i\omega t}\sin\theta[l(l+1)P_l(\cos\theta) + 2\cot\theta \partial_\theta P_l(\cos\theta)], \quad (6.6)$$

$$C_{t\phi} = \frac{\Lambda}{12}(l-1)(l+2)Re^{-i\omega t}\sin\theta \partial_\theta P_l(\cos\theta). \quad (6.7)$$

As such, they resemble the perturbations of the energy-momentum tensor for axial perturbations. In fact, choosing the overall constant

$$R = \frac{2i}{\omega}I_1, \quad (6.8)$$

the identification is exact provided that the energy-momentum tensor of axial perturbations is evaluated at $I_0 = 0$, in which case the corresponding boundary metric is conformally flat,

$$ds_{\mathcal{J}}^2(\text{axial}) = -dt^2 - \frac{3}{\Lambda}(d\theta^2 + \sin^2\theta d\phi^2). \quad (6.9)$$

Thus, using the dual boundary metrics (6.5) and (6.9), it follows that

$$C_{ab}(\text{polar}) = \kappa^2 \delta T_{ab}(\text{axial}) \quad (6.10)$$

for the supersymmetric partner boundary conditions

$$R = i\omega_s J_0, \quad I_0 = 0 \quad (6.11)$$

respectively, so that ω stays the same on both sides of the equality.

(ii) Axial perturbations: next, we consider the boundary metric for axial perturbations of AdS_4 black holes, which is given in general by

$$ds_{\mathcal{J}}^2(\text{axial}) = -dt^2 - \frac{3}{\Lambda}(d\theta^2 + \sin^2\theta d\phi^2) + 2\frac{iI_0}{\omega}e^{-i\omega t}\sin\theta \partial_\theta P_l(\cos\theta) dt d\phi. \quad (6.12)$$

In this case, the Cotton tensor of the metric has the following non-vanishing components

$$C_{tt} = -\frac{2m\Lambda^2}{3\omega}\omega_s I_0 e^{-i\omega t}P_l(\cos\theta), \quad (6.13)$$

$$C_{\theta\theta} = \frac{i\Lambda}{6\omega}l(l+1)I_0 \left(1 + \frac{3\omega^2}{\Lambda}\right) e^{-i\omega t}P_l(\cos\theta) + \frac{i\Lambda}{6\omega}I_0 \left(l(l+1) + \frac{6\omega^2}{\Lambda}\right) e^{-i\omega t}\cot\theta \partial_\theta P_l(\cos\theta), \quad (6.14)$$

$$C_{\phi\phi} = -\frac{i\Lambda}{6\omega}l(l+1)I_0 \left(l(l+1) - 1 + \frac{3\omega^2}{\Lambda}\right) e^{-i\omega t}\sin^2\theta P_l(\cos\theta) - \frac{i\Lambda}{6\omega}I_0 \left(l(l+1) + \frac{6\omega^2}{\Lambda}\right) e^{-i\omega t}\sin\theta\cos\theta \partial_\theta P_l(\cos\theta), \quad (6.15)$$

$$C_{t\theta} = \frac{\Lambda}{6}(l-1)(l+2)I_0 e^{-i\omega t}\partial_\theta P_l(\cos\theta), \quad (6.16)$$

which resemble the perturbations of the energy-momentum tensor for polar perturbations. The identification becomes exact choosing

$$I_0 = \frac{3i\omega}{2\Lambda} J_0, \tag{6.17}$$

provided that $R = 0$, in which case the corresponding boundary metric is conformally flat,

$$ds_{\mathcal{S}}^2(\text{polar}) = -dt^2 - \frac{3}{\Lambda}(d\theta^2 + \sin^2\theta d\phi^2). \tag{6.18}$$

Thus, using the dual boundary metrics (6.12) and (6.18), it follows that

$$C_{ab}(\text{axial}) = \kappa^2 \delta T_{ab}(\text{polar}) \tag{6.19}$$

for the supersymmetric partner boundary conditions

$$R = 0, \quad \frac{I_1}{I_0} = \frac{3}{\Lambda} i\omega \left(1 - \frac{\omega}{\omega_s} \right) + \frac{6m}{(l-1)(l+2)}, \tag{6.20}$$

respectively, so that ω is the same on both sides of the equality, as before. In this case, the perturbations satisfy mixed boundary conditions on both sides of the relation.

Remarkably, the supersymmetric partner boundary conditions that realize the energy-momentum/Cotton tensor duality for AdS_4 black holes are only these ones with shear viscosity

$$\eta_{\text{axial}} = \eta_{\text{polar}}. \tag{6.21}$$

Thus, the true hydrodynamic modes of very large black holes with frequencies Ω_s and Ω_{\pm} , which fit precisely in this framework, admit a new alternative description in terms of the three-dimensional Chern-Simons gravitational action on the dual boundary. The perturbations of the Schwarzschild metric at the conformal boundary, which arise on the right-hand side of the correspondence (6.10) and (6.19), are simply zero,

$$\delta g_{\mu\nu} |_{\mathcal{S}} = 0. \tag{6.22}$$

7. Conclusions

We have computed the boundary energy-momentum tensor of AdS_4 black holes for gravitational perturbations that satisfy arbitrary boundary conditions at spatial infinity. The (yet mysterious) relation between the effective Schrödinger problems for axial and polar perturbations, which is best described by supersymmetric quantum mechanics, translates into a duality between the energy-momentum and the Cotton tensor for appropriately chosen boundary conditions at spatial infinity. This framework accommodates the hydrodynamic modes of large AdS_4 black holes, which satisfy the KSS bound $\eta/s = 1/4\pi$, and, as such, it can be viewed as a new correspondence operating on this bound.

Some related remarks have also appeared recently in the literature, [35], and in particular [36] that introduces the notion of dual gravitons on general grounds, but their manifestation in AdS_4 black hole backgrounds has not been made explicit. The results also seem

to be related to the (electric-magnetic) duality rotations of the linearized four-dimensional Einstein equations, [37] (but see also [38] for earlier work), which are formulated with no reference to Killing symmetries; for further discussion and generalizations (including Einstein equations with cosmological constant) we refer the reader to the literature [39–41]. Clearly, these connections deserve further study that is left to future work. The applications in AdS_4/CFT_3 correspondence at finite temperature in view of the proposed correspondence with the gravitational Chern-Simons theory on the dual boundary will also be investigated in detail in separate publication.

Finally, another interesting question that emerged in this context is the possibility to construct exact radiative metrics of vacuum Einstein equations with negative cosmological constant, which settle to large AdS_4 black holes and account for the special frequencies of their hydrodynamic modes upon linearization. If this expectation materializes, the hydrodynamic modes will be extended in the non-linear regime and provide the gravity dual of non-linear hydrodynamics in closed form. Embedding the Ricci flow into gravity seems to play a role in this direction and it will also be investigated further in the future.

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Note added in v2. The bulk interpretation of the energy-momentum/Cotton tensor duality was investigated further by the author in the recent paper [42]. There, it was found that spherical gravitational perturbations of AdS_4 space-time, which also split into axial and polar classes, are simply interchanged by the electric/magnetic duality of linearized gravity. In this simplified case, the axial and polar perturbations obey the same Schrödinger problem and thus the same boundary conditions at spatial infinity. The electric/magnetic duality of gravitational perturbations around AdS_4 space-time applies to all possible boundary conditions and it has holographic manifestation as energy-momentum/Cotton tensor duality at the conformal infinity.

New features arise in the presence of black holes, since the axial and polar perturbations satisfy supersymmetric partner Schrödinger problems. Also, it is not known whether the electric/magnetic duality of linearized gravity in the bulk persists for perturbations around non-trivial backgrounds, such as the AdS_4 Schwarzschild solution. However, we believe that there is a remnant of duality in the linearized theory, which explains the supersymmetric

partnership of the black hole perturbations, although it might not be applicable to all possible boundary conditions at spatial infinity. In fact, its validity might very well be restricted to the special boundary conditions singled out in the present work and provide the missing link for the bulk interpretation of the energy-momentum/Cotton tensor duality for AdS_4 black holes. In this context, the gravitational electric/magnetic duality will act as symmetry of the KSS bound, in analogy with S -duality of BPS states of gravitational theories; the same rational may also apply to the more general hydrodynamic relation $\eta_{\text{axial}} = \eta_{\text{polar}}$ under the privileged set of boundary conditions.

These problems require separate investigation, which we intend to present elsewhere to illuminate their physical interpretation.

A. Coefficients of the asymptotic expansion

In this appendix we summarize the first few coefficients in the asymptotic expansion of the metric functions arising in the perturbations of AdS_4 black holes. These are the only relevant terms for the computation of the boundary energy-momentum tensor under general boundary conditions.

(i) Axial perturbations: the coefficients of the metric function $h_0(r)$ are

$$\begin{aligned} \alpha_0 &= -\frac{i\Lambda}{3\omega}I_0, & \beta_0 &= I_0, & \gamma_0 &= -i\frac{(l-1)(l+2)}{2\omega}I_0, \\ \delta_0 &= \frac{l(l+1)}{2\Lambda}I_0 - \frac{i}{3\omega}\left((l-1)(l+2) + \frac{3\omega^2}{\Lambda}\right)I_1, \end{aligned} \tag{A.1}$$

expressing them all in terms of I_0 and I_1 for convenience. Likewise, the coefficients of the metric function $h_1(r)$ are

$$\alpha_1 = -\frac{3}{\Lambda}I_0, \quad \beta_1 = -\frac{3}{\Lambda}I_1. \tag{A.2}$$

(ii) Polar perturbations: the computations are much more involved now and the expressions are quite cumbersome. The coefficients in the asymptotic expansion of $H_0(r)$ turn out to be

$$A_0 = \left(2i(\omega_s + \omega) - \frac{4m\Lambda}{(l-1)(l+2)} + \frac{\omega^2}{m\Lambda}(l-1)(l+2)\right)J_0 - \frac{2\Lambda}{3}J_1, \tag{A.3}$$

$$\begin{aligned} B_0 &= (l-1)(l+2)\left(1 + \frac{i\omega}{m\Lambda}\left[1 - \frac{12m^2\Lambda}{(l-1)^2(l+2)^2}\right]\right)J_0 + \\ &+ \frac{(l-1)(l+2)}{6m}\left((l-1)(l+2) + \frac{6\omega^2}{\Lambda} - \frac{12i\omega m}{(l-1)(l+2)}\right)J_1, \end{aligned} \tag{A.4}$$

$$\begin{aligned} C_0 &= \left(-\frac{(l-1)(l+2)}{4m\Lambda}\left[l(l+1)(l(l+1)-4) + \frac{6\omega^2}{\Lambda}(l-1)(l+2)\right] + \right. \\ &+ \left.\frac{6m}{(l-1)(l+2)}\left[l(l+1)-4 + \frac{6\omega^2}{\Lambda}\right] + \frac{6i\omega}{\Lambda}\right)J_0 + \\ &+ \left(l(l+1)-4 + \frac{6\omega^2}{\Lambda} + \frac{i\omega}{2m\Lambda}(l-1)(l+2)\left[(l-1)(l+2) + \frac{6\omega^2}{\Lambda}\right]\right)J_1. \end{aligned} \tag{A.5}$$

Likewise, the coefficients in the asymptotic expansion of $H_1(r)$ are given by

$$A_1 = \left(i\omega - \frac{2m\Lambda}{(l-1)(l+2)} \right) J_0 - \frac{\Lambda}{3} J_1, \quad (\text{A.6})$$

$$B_1 = \left(l(l+1) - 1 - \frac{12m^2\Lambda}{(l-1)^2(l+2)^2} \right) J_0 - \left(i\omega + \frac{2m\Lambda}{(l-1)(l+2)} \right) J_1, \quad (\text{A.7})$$

$$C_1 = 3 \left(m \frac{l(l+1) - 4}{(l-1)(l+2)} + \frac{i\omega}{2\Lambda} \left[l(l+1) + 2 - \frac{24m^2\Lambda}{(l-1)^2(l+2)^2} \right] \right) J_0 + \frac{1}{2} \left(l(l+1) - 4 - \frac{6i\omega}{\Lambda} \left[i\omega + \frac{2m\Lambda}{(l-1)(l+2)} \right] \right) J_1. \quad (\text{A.8})$$

Finally, the coefficients in the asymptotic expansion of $K(r)$ are

$$R = -A_1, \quad B = \frac{3i\omega}{\Lambda} A, \quad (\text{A.9})$$

$$A = -\frac{1}{2} \left(l(l+1) - \frac{24m^2\Lambda}{(l-1)^2(l+2)^2} \right) J_0 + \left(i\omega + \frac{2m\Lambda}{(l-1)(l+2)} \right) J_1, \quad (\text{A.10})$$

$$C = -\frac{1}{4\Lambda} \left(l(l+1) \left[l(l+1) - \frac{12\omega^2}{\Lambda} - \frac{24m^2\Lambda}{(l-1)^2(l+2)^2} \right] + 12i\omega m \left[1 - \frac{24i\omega m}{(l-1)^2(l+2)^2} \right] \right) J_0 + \left(m \frac{l(l+1)}{(l-1)(l+2)} + \frac{i\omega}{\Lambda} \left[1 - \frac{6\omega^2}{\Lambda} + \frac{12i\omega m}{(l-1)(l+2)} \right] \right) J_1. \quad (\text{A.11})$$

In all expressions above the results are described entirely in terms of the coefficients J_0 and J_1 for convenience, although this does not particularly simplify the lengthy formulae.

B. Energy-momentum tensor on ∂M_r

In this appendix we provide the intermediate results in the calculation of the boundary energy-momentum tensor for perturbations of AdS_4 black holes.

(i) Axial perturbations: holographic renormalization yields the following result for the perturbations of the energy-momentum tensor on ∂M_r in terms of the corresponding metric coefficients $h_0(r)$ and $h_1(r)$,

$$\kappa^2 \delta T_{t\phi} = \left(\left[\frac{2}{r} \sqrt{f(r)} + \frac{f'(r)}{2\sqrt{f(r)}} - 2\sqrt{-\frac{\Lambda}{3}} + \sqrt{-\frac{3}{\Lambda} \frac{(l-1)(l+2)}{2r^2}} \right] h_0(r) - \frac{1}{2} \sqrt{f(r)} (h_0'(r) + i\omega h_1(r)) \right) e^{-i\omega t} \sin\theta \partial_\theta P_l(\cos\theta), \quad (\text{B.1})$$

$$\kappa^2 \delta T_{\theta\phi} = -\frac{1}{2} \left(\sqrt{f(r)} h_1(r) + i\omega \sqrt{-\frac{3}{\Lambda} \frac{h_0(r)}{f(r)}} \right) e^{-i\omega t} \times \sin\theta [l(l+1) P_l(\cos\theta) + 2\cot\theta \partial_\theta P_l(\cos\theta)], \quad (\text{B.2})$$

whereas the other components vanish.

(ii) Polar perturbations: likewise, we obtain the following result for the perturbations of the energy-momentum tensor on ∂M_r in terms of the corresponding metric functions $H_0(r)$, $H_1(r)$ and $K(r)$,

$$\begin{aligned} \kappa^2 \delta T_{tt} = f(r) & \left(\left[\frac{3}{r} \sqrt{f(r)} - 2\sqrt{-\frac{\Lambda}{3}} - \frac{1}{r^2} \sqrt{-\frac{3}{\Lambda}} \right] H_0(r) - \sqrt{f(r)} K'(r) \right. \\ & \left. + \frac{(l-1)(l+2)}{2r^2} \sqrt{-\frac{3}{\Lambda}} K(r) \right) e^{-i\omega t} P_l(\cos\theta), \end{aligned} \quad (\text{B.3})$$

$$\begin{aligned} \kappa^2 \delta T_{\theta\theta} = & \left(\left[r\sqrt{f(r)} + \frac{r^2 f'(r)}{2\sqrt{f(r)}} - 2r^2 \sqrt{-\frac{\Lambda}{3}} + \frac{\omega^2 r^2}{2f(r)} \sqrt{-\frac{3}{\Lambda}} \right] K(r) \right. \\ & + \frac{r^2}{2} \sqrt{f(r)} K'(r) - i\omega \frac{r^2}{\sqrt{f(r)}} H_1(r) - \frac{r^2}{2} \sqrt{f(r)} H_0'(r) \\ & - \frac{r}{2} \left[\sqrt{f(r)} + \frac{r f'(r)}{2\sqrt{f(r)}} \right] H_0(r) \left. \right) e^{-i\omega t} P_l(\cos\theta) \\ & - \frac{1}{2} \sqrt{-\frac{3}{\Lambda}} H_0(r) e^{-i\omega t} \cot\theta \partial_\theta P_l(\cos\theta), \end{aligned} \quad (\text{B.4})$$

$$\begin{aligned} \kappa^2 \frac{\delta T_{\phi\phi}}{\sin^2\theta} = & \left(\left[r\sqrt{f(r)} + \frac{r^2 f'(r)}{2\sqrt{f(r)}} - 2r^2 \sqrt{-\frac{\Lambda}{3}} + \frac{\omega^2 r^2}{2f(r)} \sqrt{-\frac{3}{\Lambda}} \right] K(r) \right. \\ & + \frac{r^2}{2} \sqrt{f(r)} K'(r) - i\omega \frac{r^2}{\sqrt{f(r)}} H_1(r) - \frac{r^2}{2} \sqrt{f(r)} H_0'(r) \\ & - \frac{r}{2} \left[\sqrt{f(r)} + \frac{r f'(r)}{2\sqrt{f(r)}} - \frac{l(l+1)}{r} \sqrt{-\frac{3}{\Lambda}} \right] H_0(r) \left. \right) e^{-i\omega t} P_l(\cos\theta) \\ & + \frac{1}{2} \sqrt{-\frac{3}{\Lambda}} H_0(r) e^{-i\omega t} \cot\theta \partial_\theta P_l(\cos\theta), \end{aligned} \quad (\text{B.5})$$

$$\kappa^2 \delta T_{t\theta} = \frac{1}{2} \left(i\omega \sqrt{-\frac{3}{\Lambda}} K(r) + \sqrt{f(r)} H_1(r) \right) e^{-i\omega t} \partial_\theta P_l(\cos\theta). \quad (\text{B.6})$$

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